Solving for d_1 and $F-d_1$

$$d_1 = \frac{2F}{3} \pm \frac{\sqrt{6K_{ij}^2 - 2F^2}}{3}$$

$$F - d_1 = \frac{F}{3} \mp \frac{\sqrt{6K_{ij}^2 - 2F^2}}{3}. \tag{39}$$

By substitution in (38)

$$8K_{ij}^6 + 8K_{ij}^4F^2 = 3F^6. \tag{40}$$

Upon solving

$$K_{ij} = \frac{F}{\sqrt{2}} \tag{41}$$

and

$$K_{ij}^2 = \frac{F^2}{2} \quad \text{where} \quad F = \frac{1}{Q_L}. \tag{42}$$

Upon substituting into the equation for normalized susceptance of an iris in circular waveguide, (14) is obtained.

$$\frac{B}{Y_0} = \frac{2}{\pi K_{ij}}\left(\frac{\lambda}{\lambda_g'}\right)^2 \tag{43}$$

whence

$$\frac{B}{Y_0} = \frac{2\sqrt{2}}{\pi} Q_L \left(\frac{\lambda}{\lambda_g'}\right)^2. \tag{44}$$

Appendix III

The window coupling factor defined by Nelson[1] for a circular iris in a plane transverse to the direction of propagation in a circular guide is

$$Q_{w2} = \left[\frac{\pi}{2}\left(\frac{\lambda_g'}{\lambda}\right)^2\right]^2 \left[\frac{B}{Y_0}\right]^2. \tag{45}$$

The diameter of a centered circular iris hole in circular waveguide is determined from the expression of a normalized shunt susceptance of a circular aperture in the transverse plane of the circular waveguide,[9] thus

$$\frac{B}{Y_0} = \frac{\lambda_g'}{2D}\left(\frac{D^2}{8.4M} - 2.344\right) \tag{46}$$

where M, the magnetic polarizability factor, is equal to $d_2^3/6$ for a centered circular aperture of diameter d_2.

Solving (45) and (46) assuming $D^2/8.4M \gg 2.344$ for values of $d_2 < 0.5$ inch

$$d_2^3 = \frac{0.561\lambda_g'^3 D}{\lambda^2\sqrt{Q_{w2}}}. \tag{47}$$

Power Transmission Through General Uniform Waveguides*

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Summary—**The problem of transmitting electromagnetic power through a dissipative waveguide in an optimal fashion is examined. At any frequency the problem is reduced to a weighted eigenvalue problem in which the maximum efficiency appears as the eigenvalue and the required excitation is specified by the corresponding eigenvector. Numerical results are presented.**

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I. Introduction

THIS PAPER examines the transmission of electromagnetic power through a dissipative uniform waveguide simultaneously utilizing a number of the proper modes of the guide. At any frequency, the power dissipated within the guide is a function of the relative amplitude and phase of the various modal constituents. The problem of transmitting in an optimal fashion is reduced to a weighted eigenvalue problem in which the maximum efficiency occurs as an eigenvalue of the required distribution of the excitation among the various modes as the corresponding eigenvector. Numerical results are presented for the case of a circular waveguide. While the form of these numerical results is certainly interesting, the gain relative to conventional single-mode transmission is slight.

Physically the problem considered for waveguide is closely related to the Borrmann effect in transmission of X rays through crystals.[1] Both may be understood in terms of destructive interference of fields at just those points at which dissipative matter is localized: herein the guide walls. In cavity measurements of dielectric constant at microwave frequencies, for example, the same interference phenomena may also produce significant departures from results of elementary transmission line calculations.[2]

The fundamental electromagnetic considerations are conveniently developed with the aid of the 6-vector operator formulation of the electromagnetic field equations as given by Bresler, Joshi and Marcuvitz.[3] The conventions and notation introduced by them will also be found convenient for purposes of this paper. In the interest of brevity these will not be repeated.

The electromagnetic field within the guide is to consist of a superposition of modes, $\mathbf{\Phi}_\alpha \rightarrow \{\boldsymbol{E}, i\boldsymbol{H}\}$ and adjoint modes $\mathbf{\Phi}_\alpha^+$, normalized to $N_\alpha = (\mathbf{\Phi}_\alpha^+, \Gamma_z \mathbf{\Phi}_\alpha)$,

$$\mathbf{\Psi}(r) = \sum C_\alpha(z)\mathbf{\Phi}_\alpha(\mathbf{p}); \tag{1}$$

$$C_\alpha(z) = C_\alpha(z_0)\epsilon^{+j\kappa_\alpha(z-z_0)}, \tag{2}$$

where the $C_\alpha(z_0)$ are Fourier coefficients

$$N_\alpha C_\alpha = (\mathbf{\Phi}_\alpha^+, \Gamma_z \mathbf{\Psi}) \tag{3}$$

determined at $z = z_0$. In view of the representation (1) a scattering transfer-matrix formalism for M modes may be introduced

$$\underline{\Psi}_s(z) = T_s(z, z_0)\underline{\Psi}_s(z_0), \tag{4}$$

in which

$$\underline{\Psi}_s(z) = [C_\alpha(z)]; \tag{5}$$

$$T_s(z, z_0) = \exp\{i\kappa(z - z_0)\}; \tag{6}$$

$$\kappa = \text{diag}\{\kappa_\alpha\}, \qquad \alpha = 1, 2, \cdots, M. \tag{7}$$

II. The Power-Associated Quadratic Form

The net-time-average power flow in the direction $\boldsymbol{z}^0$ across a plane transverse to the guide axis associated with a field $\mathbf{\Psi}(z)$ is

$$P_{\text{av}}(z) = \tfrac{1}{2}(\mathbf{\Psi}, \Gamma_z \mathbf{\Psi}), \tag{8}$$

the real part of the integral of the complex Poynting vector over the guide cross section at z. In terms of the transfer formalism (4) and the (Hermitian, matrix) inner product (8) becomes

$$P_{\text{av}}(z) = (\underline{\Psi}_s, \underline{p}_s \underline{\Psi}_s) \tag{9}$$

provided the elements of the $M \times M$ matrix, $\underline{p}_s = [p_{\alpha\beta}]$, are defined as

$$p_{\alpha\beta} = \tfrac{1}{2}(\mathbf{\Phi}_\alpha, \Gamma_z \mathbf{\Phi}_\beta). \tag{10}$$

From the definition (10) and the Hermitian character of $\Gamma_z = \Gamma_z^+$, it follows that $\underline{p}_s = \underline{p}_s^+$. The simplification for conventional lossless waveguides, for which the modal problem is self-adjoint, $\mathbf{\Phi}_\alpha^+ = \mathbf{\Phi}_\alpha$, is due to the fact that (10) then coincides with the orthogonality relations and $p_{\alpha\beta} = N_\alpha \delta_{\alpha\beta}$.

If representations, $\underline{\Psi}$, alternative to the scattering representation (4) are introduced through

$$\underline{\Psi}_s = U\underline{\Psi}, \tag{11}$$

where U is a nonsingular transformation, the appropriate weight operator p follows from the necessary invariance of (9):

$$\underline{p} = U^+\underline{p}_s U. \tag{12}$$

Note $\underline{p}$ remains Hermitian. Subsequently the subscript s will be dropped when the particular representation is irrelevant.

Special simplifications in the form of $\underline{p}$ result from any symmetries the waveguide may possess in addition to the translational symmetry assumed initially. The most important such symmetry is invariance under reflection in any plane transverse to the guide axis. The $M = 2N$ modes then occur in natural pairs, $\mathbf{\Phi}_\alpha$ and its mirror image designated $\mathbf{\Phi}_{N+\alpha}$, with propagation constants κ_α and $-\kappa_\alpha$, respectively. The matrix $\underline{p}_s$ therefore has the appearance

$$\left[\begin{array}{c|c} \mathcal{A} & i\mathcal{B} \\ \hline -i\mathcal{B} & -\mathcal{A} \end{array}\right] \tag{13}$$

in which $\mathcal{A}$ and $\mathcal{B}$ are Hermitian $N \times N$ submatrices. The circular waveguide, employed later for numerical results, also has rotational symmetries. A particular consequence is that all terms in $\underline{p}$ coupling the low-loss H_{0n} (TE_{0n}) modes to the remaining mode types of the guide vanish.

III. Optimal Transmission

Consider the utilization of a section of dissipative waveguide for the transmission of power in such manner that the fraction of the input power absorbed by the guide is minimized. An efficiency may be defined by the power ratio

$$\eta(z, z_0) = \frac{P_{\text{av}}(z)}{P_{\text{av}}(z_0)} = \frac{(T\underline{\Psi}, \underline{p}T\underline{\Psi})}{(\underline{\Psi}, \underline{p}\underline{\Psi})} \tag{14}$$

[1] E. Mayer, "Absorption of X-Rays in Perfect Crystals," Ph.D dissertation (Physics), Polytechnic Inst. of Brooklyn, N. Y.; June, 1952. See also G. Borrmann, "Über Extinktions-diagramme von Quarz," *Phys. Z.* vol. 42, pp. 157–162; July 15, 1941; H. Cole, F. W. Chambers and C. G. Wood, "X-ray polarizer," *J. Appl. Phys.* vol. 32, pp. 1942–1945; October, 1961.

[2] G. Persky and H. M. Altschuler, "Dissipation in Uniform (Dielectric Filled) Waveguides," Microwave Res. Inst., Polytechnic Inst. of Brooklyn, N. Y., Electrophysics Memo. 69, PIBMRI-946-61; September, 1961.

[3] A. D. Bresler, G. H. Joshi and N. Marcuvitz, "Orthogonality properties for modes in passive and active uniform waveguides," *J. Appl. Phys.*, vol. 29, pp. 794–799; May, 1958.

wherein it is understood that $\underline{\Psi}=\underline{\Psi}(z_0)$ and $T=T(z, z_0)$; clearly, η may take on any real value. Now let $\underline{\Psi}$ be constrained so that

$$(\underline{\Psi}, p\underline{\Psi}) = 1, \tag{15}$$

then minimum loss corresponds to the maximum value of η, and for passive guides $\eta \leq 1$.

The excitations, subject to the constraint (15), about which η is stationary are among the solutions of

$$\delta\{(\underline{\Psi}, T^+pT\underline{\Psi}) - \theta[(\underline{\Psi}, p\underline{\Psi}) - 1]\} = 0 \tag{16}$$

without special constraints. These particular excitations are solutions of the weighted eigenvalue problem

$$[T^+pT - \theta_\gamma p]\underline{\Theta}_\gamma = 0 \tag{17}$$

which may be normalized to satisfy (15). Substitution into (14) yields $\eta=\theta_\gamma$. Hence the maximum such eigenvalue θ_γ is the maximum efficiency, and the corresponding eigenvector proportional to the required excitation.

It is clear that only real eigenvalues can be interpreted as efficiencies, and the eigenvalues θ_γ were tacitly assumed real. All eigenvalues for which the corresponding eigenvector may be normalized as required are indeed real, for

$$(T^+pT\underline{\Theta}_\gamma, \underline{\Theta}_\gamma) - (\underline{\Theta}_\gamma, T^+pT\underline{\Theta}_\gamma) = 0 \tag{18a}$$

$$(\theta_\gamma^* - \theta_\gamma)(\underline{\Theta}_\gamma, p\underline{\Theta}_\gamma) = 0. \tag{18b}$$

This formalism has also been employed in connection with conventional $2N$-port networks, and additional interesting results applicable to this special case have been obtained.[4,5]

IV. Numerical Results

The general theory of the preceding sections was applied to circular waveguide. This type of guide has presently aroused wide interest in connection with low-loss transmission via the circular electric, *i.e.*, the H_{0n} modes. Dissipation was introduced through specification of a complex surface impedance boundary condition at the guide wall, radius a,

$$\boldsymbol{E} = Z \cdot \boldsymbol{H} \times \nu^0. \tag{19}$$

Correspondence with the metallic waveguide is established on setting

$$Z = \sqrt{\frac{\mu_0}{\epsilon_0}}\, \frac{k\delta(1-i)}{2}, \qquad \delta = \sqrt{\frac{2}{\mu_0\sigma\omega}}, \tag{20}$$

where $k=\omega\sqrt{\mu_0\epsilon_0}$ and σ is the conductivity (ohm-cm)$^{-1}$ of the guide walls. Only selected results are given here; for details of calculation as well as additional computations reference may be had to the original dissertation.[5]

As was previously indicated, due to the symmetries of the guide the H_{0n} modes may be treated independently of any other mode type. The general nature of the results are illustrated by Figs. 1 and 2. In Fig. 1 the efficiency in optimal transmission utilizing the first two such modes ($M=2$), propagating in the $+z$ direction, η_μ, is compared with conventional transmission utilizing only the first of these modes η_1. The ratio

$$\lambda(z, z_0) = \frac{\eta_\mu(z, z_0) - \eta_1(z, z_0)}{\eta_\mu(z, z_0)} \tag{21}$$

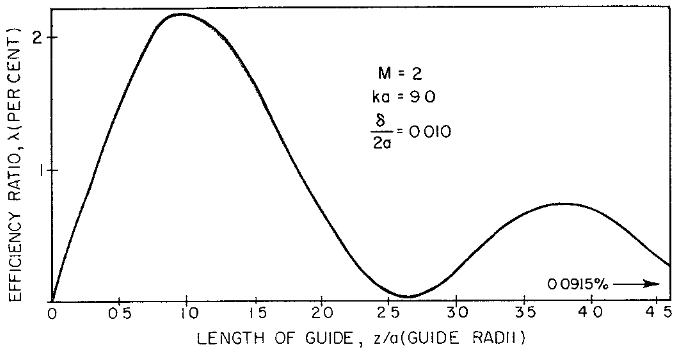

Fig. 1—Relative efficiency of optimal direct-wave transmission λ.

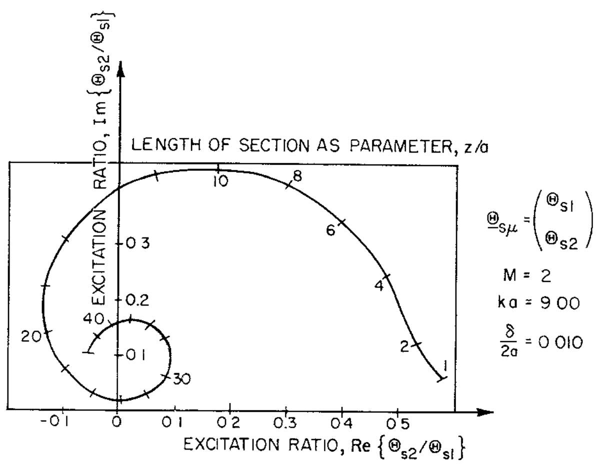

Fig. 2—Excitation in optimal direct-wave transmission, $H_{s\mu}$.

TABLE I

ka	Asymptotic Efficiency Ratio, λ(∞)	
	$M=2$	$M=3$
9	0.091	
14	0.088	0.118
16	0.090	0.118
18	0.088	0.115
20	0.089	0.117

[4] L. B. Felsen and W. K. Kahn, *Proc. Symp. on Millimeter Waves*, Polytechnic Inst. of Brooklyn, N. Y.; March 31, April 1, 2, 1959.

[5] W. K. Kahn, "A Theoretical and Experimental Investigation in Multimode Network and Waveguide Transmission," D.E.E. dissertation, Polytechnic Inst. of Brooklyn, N. Y.; June, 1960.

plotted as a function of $(z-z_0)/a$ for the values $ka=9.0$ and $\delta/2a=0.010$. Even for this rather dissipative guide the improvement is seen to be slight. The asymptotic value of $\lambda(\infty)$ approached as $(z-z_0)/a\rightarrow\infty$ is indicated by the arrow at the lower right. In Fig. 2 the ratio of the two components of the excitation $\underline{\Psi}_s(z_0)=\underline{\Theta}_{s,\mu}$ are plotted in the complex plane with $(z-z_0)/a$ as a parameter for the same values of ka and $\delta/2a$. For short lengths of guide the ratio λ is increased by a factor of about 5 if the reflected waves are also optimally adjusted, *i.e.*, $M=2N=4$ modes (two waves propagating in the $+z$ and two in the $-z$ direction) are employed.

The asymptotic value of λ has been computed for two and three modes propagating in the $+z$ direction for several values of ka. There are listed in Table I.

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Further Considerations on Fabry-Perot Type Resonators*

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Summary—**An integral equation valid for Fabry-Perot type resonators with reflectors of arbitrary curvature and spacing is derived, and equations for the planar, confocal, and spherical geometries are considered further. A numerical iteration method is used to solve the equations, and the properties of the various solutions for the different kernels are discussed. Results show that the confocal type has the lowest diffraction loss, and that the losses in the planar- and spherical-type geometries are identical, as are the normal mode field distributions over the reflectors, apart from a change in sign of the phase angle. Variational methods are applied to give results for the eigenvalues of the planar geometry with great facility, particularly for cases where the eigenvalues are closely spaced. Some potential uses and the respective merits of the resonators are briefly mentioned.**

I. Introduction

FREE-SPACE resonators, analogous to the optical Fabry-Perot interferometer, continue to play a dominant role in measurements and physical devices for very short microwaves, and also in the new devices for producing coherent light [1]–[4]. Previous work has discussed the application of this interferometer to millimeter wavelengths [5], [6], an important result being that coupling to such resonators could be effected by a whole series, or grating, of coupling holes over the area of the metallic reflector. Such a method can obviously be applied to reflectors of arbitrary shape [7], and is most useful for very short microwaves where optical methods, such as multilayer dielectric films, are not easy to apply. The planar type of reflector system, due to the absence of mode degeneracy, possesses some advantages in routine measurements of wavelength and dielectric constants [8]. Diffraction losses, though larger for given dimensions than those of the confocal-type resonator [9], can still be made small at the shorter wavelengths, and their effect on measurements reduced. However, for a given wavelength and reflector size, such losses do limit the Q value obtainable, and for some purposes such as filter applications, and threshold conditions in lasers, the confocal type may be preferable. However, the planar geometry, though more critical in adjustment and in the degree of flatness required, readily permits single-mode operation, and potentially gives a larger power output than the confocal.

One of the difficulties in evaluating the quality of these free-space resonators is that of diffraction. This leads to diffraction losses and to phase changes which differ slightly from those corresponding to plane wave propagation. The application of integral equations for the solution of such problems was indicated by Goubau and Schwering in their work on the guided propagation of electromagnetic wave beams [10], [11]. Fox and Li [12] also considered various resonator types, and set up the integral equations using the Huygens-Kirchhoff diffraction theory. Numerical solutions for the eigenvalues and eigenfunctions, or field distribution, were obtained by computing the steady state reached after a large number of bounces between the reflectors. Boyd and Gordon [13] also considered the confocal type resonator in some detail. This arrangement is somewhat unique

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